

**Anomalous translational velocity of vortex ring with finite-amplitude Kelvin waves**C. F. Barenghi,<sup>1</sup> R. Hänninen,<sup>2</sup> and M. Tsubota<sup>2</sup><sup>1</sup>*School of Mathematics, University of Newcastle, Newcastle NE1 7RU, United Kingdom*<sup>2</sup>*Department of Physics, Osaka City University, Sugimoto 3-3-138, 558-8585 Osaka, Japan*

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We consider finite-amplitude Kelvin waves on an inviscid vortex assuming that the vortex core has infinitesimal thickness. By numerically solving the governing Biot-Savart equation of motion, we study how the frequency of the Kelvin waves and the velocity of the perturbed ring depend on the Kelvin wave amplitude. In particular, we show that, if the amplitude of the Kelvin waves is sufficiently large, the perturbed vortex ring moves backwards.

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**I. INTRODUCTION**

Vortex rings are among the most important and most studied objects of fluid mechanics [1,2]. It has been known since the times of Lord Kelvin [3] that a vortex ring is subject to wavy distortions (sinusoidal displacements of the vortex core) called Kelvin waves [4]. In the case of viscous vortex rings, the stability of these waves is a problem with subtle aspects [5] which are still the focus of intense mathematical scrutiny [6]. Our concern is the simpler case in which the fluid is inviscid and the vortex core has infinitesimal thickness. This case refers to the idealized context of classical Euler fluids, but is realistic for superfluids, which have zero viscosity and microscopic vortex core thickness.

Vortex rings have indeed been central to superfluidity [7] since the pioneering experiments on the nucleation of quantized vorticity by moving ions [8], the early investigations into rotons as ghosts of vanished vortex rings [9], and the nature of the superfluid transition [10]. The current interest in superfluid vortex rings extends to the physics of cold atomic gases [11] and the discovery of new nonlinear solutions [12] of the Gross-Pitaevskii's nonlinear Schrödinger equation (NLSE) for a Bose-Einstein condensate. Vortex rings are also important in the study of superfluid turbulence [13]. For example, they have been used as tools to study the Kelvin wave cascade [14] which is responsible for the dissipation of turbulent kinetic energy near absolute zero, and to investigate the effects of vortex reconnections [15], which are the key feature of turbulence; they are also used as simple models of the vortex loops which make up the turbulence [16].

Kelvin waves play a role in all examples listed above. The dispersion relation of Kelvin waves of infinitesimal amplitude  $A$  on a circular vortex ring of given radius  $R$ , circulation  $\kappa$ , and vortex core radius  $a$  is [17]

$$\omega = \frac{\kappa}{2\pi a^2} \left( 1 - \sqrt{1 + ka \frac{K_0(ka)}{K_1(ka)}} \right), \quad (1)$$

where  $\omega$  is the angular velocity of the wave and  $k$  the wave number. Functions  $K_n(x)$  are modified Bessel functions of order  $n$ . The above dispersion relation is also valid for waves on a straight vortex [18]. The properties of small-amplitude Kelvin waves have been already investigated [19], but little is known of what happens at large wave amplitude. The sta-

bility problem becomes nonlinear, hence more difficult, and a numerical approach is necessary.

Recently, an astonishing prediction was made by Kiknadze and Mamaladze [20] that, at sufficiently large amplitude, the perturbed vortex ring moves backwards. Unfortunately the prediction arises from numerical analysis based on the local induction approximation (LIA) to the exact equation of motion, which is the Biot-Savart law (BSL). The advantage of the LIA over the BSL is that it is analytically simpler and computationally cheaper. If  $N$  is the number of discretization points along a vortex filament, the cost of the computation grows as  $N$  under the LIA, whereas under the BSL it grows as  $N^2$ . The use of the LIA was pioneered by Schwarz [22] in his numerical studies of homogeneous isotropic turbulence. His results obtained using the LIA compared reasonably well with results obtained using the BSL, because long-range effects tend to cancel out in the isotropic vortex configurations which he considered. In less isotropic cases, however, for example in rotating turbulence [23], the LIA may not be a good approximation. In particular, the LIA yields wrong predictions about the stability and motion of vortex knots [24], structures which are geometrically similar to (although topological different from) the perturbed vortex rings considered by Kiknadze and Mamaladze [20].

Our first aim is thus to use the exact BSL to investigate the claim of Kiknadze and Mamaladze that the perturbed vortex ring can move backwards [20]. Our second aim is to carry out a more detailed examination of the effects of large-amplitude Kelvin waves on the motion of a vortex ring.

**II. MODEL**

Our approach is based on the vortex filament model of Schwarz [22] which is appropriate to superfluid helium due to the smallness of the vortex core radius  $a$  compared to the radius of the vortex ring  $R$ . Essentially, a vortex is treated as a topological line defect, that is to say a curve in three-dimensional space. In the absence of dissipation (zero temperature), the vortex at the point  $\mathbf{r}$  moves with velocity  $d\mathbf{r}/dt = \mathbf{v}_L$  where  $\mathbf{v}_L$  is equal to the local superfluid velocity  $\mathbf{v}_s$  that is given by the following Biot-Savart line integral calculated along the entire vortex configuration

$$\mathbf{v}_s(\mathbf{r}, t) = \frac{\kappa}{4\pi} \oint \frac{(\mathbf{s} - \mathbf{r}) \times d\mathbf{s}}{|\mathbf{s} - \mathbf{r}|^3}. \quad (2)$$

Here  $\mathbf{s}$  denotes a variable location along the vortex filament. To implement the BSL, the vortex configuration is discretized into a large number of segments. The technique to handle the singularity that one meets when one tries to evaluate the integral at those discrete points that are used to describe the vortex line can be avoided by splitting the integral into local and nonlocal parts [22]. The velocity of a point  $\mathbf{s}$  on the vortex is thus

$$\mathbf{v}_L = \frac{\kappa}{4\pi} \mathbf{s}' \times \mathbf{s}'' \ln\left(\frac{2\sqrt{l_+ l_-}}{e^{1/2} a}\right) + \frac{\kappa}{4\pi} \oint' \frac{(\mathbf{s}_1 - \mathbf{s}) \times d\mathbf{s}_1}{|\mathbf{s}_1 - \mathbf{s}|^3}, \quad (3)$$

where  $\xi$  is the arc length, the vectors  $\mathbf{s}' = d\mathbf{s}/d\xi$ ,  $\mathbf{s}'' = d^2\mathbf{s}/d\xi^2$  are, respectively, the local tangent and the local normal to the vortex at the point  $\mathbf{s}$ . The quantities  $l_-$  and  $l_+$  are the lengths of the line segments connected to the discretization point  $\mathbf{s}$  and the prime above the integral symbol means that the line integration now extends only along the remaining vortex segments. One should note that we use a hollow core vortex, which results that the scaling factor in front of  $a$  in Eq. (3) is  $\exp(1/2)$  rather than  $\exp(1/4)$  which is for a solid rotating core and appears in a paper by Schwarz [22]. The recent progress in using the Gross-Pitaevskii nonlinear Schrödinger equation for quantum fluids suggests that the hollow core model should be more appropriate [25]. The exact value of the core size is not important here. What matters is that it is orders of magnitudes smaller than the radius of the ring or the amplitude of the waves, so that we can use the concept of vortex filament. For example, in a typical helium turbulence experiment the measured vortex line density  $L$  is  $10^4$  or  $10^6$   $\text{cm}^{-2}$ , which means that the intervortex spacing is  $1/\sqrt{L} = 0.01$  or  $0.001$  cm, which is a million or hundred thousands times bigger than the vortex core radius ( $10^{-8}$  cm) in  $^4\text{He}$ .

The local induction approximation (LIA) is obtained by neglecting the nonlocal part and is typically written in the form

$$\mathbf{v}_L = \beta \mathbf{s}' \times \mathbf{s}'', \quad (4)$$

where  $\beta = \kappa \ln(c\langle R \rangle/a)/4\pi$ ,  $\langle R \rangle$  is some average curvature, and  $c$  is of order unit; the last two parameters are adjusted to obtain better agreement with full nonlocal calculations. By choosing  $c = 8 \exp(-1/2)$  and  $\langle R \rangle$  to be the local radius of curvature one obtains fairly good results and additionally a limit that gives correctly the velocity for the perfect ring.

The calculation of the kinetic energy  $E$  of the vortex would not be accurate if carried out on a three-dimensional mesh around the vortex due to rapid changes of the velocity field near the vortex core. Fortunately in our case the vortex filament forms a closed loop and the velocity field goes to zero at infinity (the calculation is performed in an infinite box), hence it is appropriate [26] to use Saffman's formula [2]

$$E = \kappa \rho_s \oint \mathbf{v}_s \cdot \mathbf{s} \times d\mathbf{s}, \quad (5)$$

where the line integration is performed along the vortex filament and  $\rho_s$  is the superfluid density.

The initial condition consists of a vortex ring of radius  $R$  with superimposed  $N$  Kelvin waves of amplitude  $A$  (that is, the wavelength of the perturbation is  $2\pi R/N$ ). Using cylindrical coordinates  $r$ ,  $\phi$ , and  $z$ , the Cartesian coordinates of the initial vortex ring are thus

$$\begin{aligned} x &= R \cos \phi + A \cos(N\phi) \cos \phi, \\ y &= R \sin \phi + A \cos(N\phi) \sin \phi, \\ z &= -A \sin(N\phi). \end{aligned} \quad (6)$$

In the absence of Kelvin waves ( $A=0$ ) the circular vortex ring moves in the positive  $z$  direction with self-induced translational speed [27]

$$v_{\text{ring}} = \frac{\kappa}{4\pi R} [\ln(8R/a) - 1/2]. \quad (7)$$

We have tested that, in the case of a circular ring, our numerical method agrees fairly well with this result.

All results presented here are obtained using ring radius  $R=0.1$  cm and values of  $a$  and  $\kappa$  which refer to  $^4\text{He}$  ( $\kappa = h/m_4 = 9.97 \times 10^{-4}$   $\text{cm}^2/\text{s}$ , where  $m_4$  is the mass of one atom, and  $a = 1.0 \times 10^{-8}$  cm). The dependence of the results on  $a$  is small, since  $a$  appears only in the slow varying logarithmic term in Eq. (3).

The numerical method to evolve the perturbed vortex ring under the BSL is based on a fourth-order Runge-Kutta scheme. The spatial discretization is typically  $\Delta\xi/R = 0.02$  and the time step  $\Delta t = 0.5 \times 10^{-3}$  s. The time step is well below the one that for a given space resolution provides stable motion of a circular vortex ring without fluctuations and resolves the oscillations of the Kelvin waves. Numerical calculations are also performed using the LIA to compare against the exact BSL.

We are unable to perform a precise stability analysis of large-amplitude Kelvin waves under the Biot-Savart Law or a stability analysis of the Runge-Kutta scheme when applied to the Biot-Savart motion—both problems are practically impossible. We find that for very large times (larger than reported in the following section) the perturbed vortex ring always breaks up at some point (that is, first deforms and later possibly attempts to reconnect with itself). We do not know whether this fate indicates an instability of the vortex for large-amplitude Kelvin waves or a numerical instability. What matters is that the lifetime of the perturbed vortex and the spatial range that it travels are much larger than the time scale of the Kelvin oscillations and the size of the ring itself, because it implies that the results which we describe are physically significant and observable in a real system.

### III. RESULTS

The first result of our numerical simulations is that Kiknadze and Mamaladze's prediction [20] obtained using

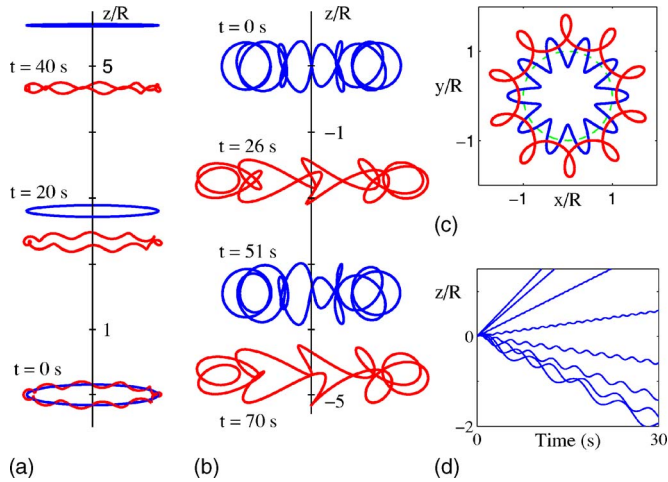


FIG. 1. (Color online) Snapshots of the vortex ring of radius  $R=0.1$  cm perturbed by  $N=10$  Kelvin waves of various amplitude  $A$  taken during the motion of the vortex. In the left panel (a) the amplitude of the Kelvin waves is small,  $A/R=0.05$ , but the perturbed vortex ring (red color) already moves slower than the unperturbed vortex (blue color). In the center panel (b) the Kelvin waves have large amplitude,  $A/R=0.35$ , and the perturbed vortex ring moves backwards (negative  $z$  direction) on average. The top right panel (c) shows the top ( $xy$ ) view of the large amplitude vortex at  $t=0$  s (blue) and  $t=26$  s (red, outermost). For comparison, a non-disturbed vortex is shown with dashed line (green). The lower right panel (d) gives the averaged location of the ring as a function of time. From top to bottom the curves correspond to  $A/R=0.0, 0.05, 0.10, \dots, 0.35$ .

the LIA is indeed correct. Integration of the motion using the exact BSL shows that, provided the amplitude of the Kelvin waves is large enough, the vortex ring moves (on the average) backwards. This result is illustrated in Figs. 1 and 2: the former shows snapshots of the ring at different times as it travels, the latter gives the average translational velocity of the ring along the  $z$  direction as a function of the amplitude  $A$  of the Kelvin waves. It is apparent that the translational velocity decreases with increasing amplitude of the Kelvin waves and can even become negative.

At some critical value of the amplitude  $A$  the translational velocity is zero and the perturbed vortex ring hovers like a stationary helicopter. In the case of  $N=10$  Kelvin waves this happens when  $A/R=0.17$  approximately, which is quite close to the LIA prediction,  $A/R=0.16$ . For  $N=6$  and  $N=20$  the critical value is, respectively,  $A/R=0.32$  and  $A/R=0.085$ . This dependence of the critical amplitude on  $N$  is in approximate agreement with the LIA prediction [20].

The backward velocity of the perturbed vortex ring depends nonlinearly on the amplitude  $A$  of the Kelvin waves. At large enough amplitude  $A$  this velocity will slow down. This can be clearly seen in Fig. 2. The Kelvin waves, that can be imagined to behave like small vortex rings, tend to turn backwards, or more precisely, on the direction opposite to the motion of the unperturbed vortex ring. The larger the amplitude the larger fraction of the ring velocity is oriented downwards. This is compensated by the decrease in velocity of the single ring, which is inversely proportional to the am-

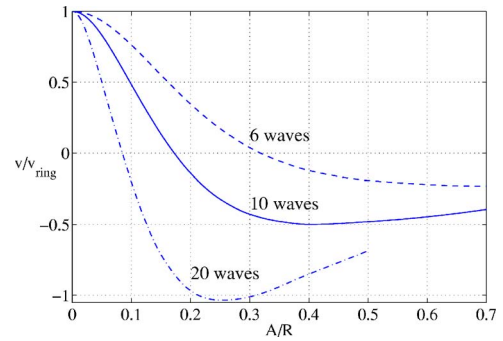


FIG. 2. (Color online) Average translational velocity of the vortex ring as a function of the initial oscillation amplitude  $A/R$ . Velocity is scaled by the velocity of the unperturbed ring,  $v_{\text{ring}}$ . The dash-dotted line corresponds to  $N=20$ , solid line to  $N=10$ , and the dashed line to  $N=6$  in Eq. (6). Critical amplitudes, above which the velocities become negative, are  $A/R=0.085$ ,  $0.17$ , and  $0.32$ , respectively.

plitude, resulting an optimum value at some amplitude. For  $N=20$  the optimum amplitude  $A \approx 0.25R$  resulting a downward velocity that is already slightly higher than the velocity upwards of the unperturbed ring.

In addition to Kelvin waves, the translational velocity of the vortex ring can be reduced by having an additional swirl velocity along the vortex core. This was considered by Wadnall, Bliss, and Zalay [21]. However, this effect does not matter in our limit of thin-core vortices, which is relevant to superfluids.

The dispersion relation of large-amplitude Kelvin waves can be obtained by tracking the motion of the vortex on the  $y=0$  plane, for example. If the amplitude  $A$  of the Kelvin wave is small, the vortex draws a circle at approximately the same angular frequency that is obtained analytically for small-amplitude Kelvin waves and given by Eq. (1). In the long wavelength limit ( $k \rightarrow 0$ ) this relation becomes

$$\omega = -\frac{\kappa k^2}{4\pi} \left[ \ln\left(\frac{2}{ka}\right) - \gamma \right], \quad (8)$$

where  $\gamma=0.5772 \dots$  is Euler's constant and the negative sign only indicates that the Kelvin waves rotate opposite to the circulation. Again the above equation differs slightly ( $-\gamma$  instead of  $1/4 - \gamma$ ) from the form given by Schwarz [22], but this is again only due to the definition of the core type.

We find that if we increase the amplitude of the Kelvin waves on the ring then the angular frequency decreases, a result which we also verified in the case of a straight vortex. Some example curves drawn by the vortex on the  $y=0$  plane are shown in Fig. 3. The average angular frequency is plotted in Fig. 4, which shows also the dispersion relation of waves on a straight vortex for comparison.

It is important to notice that, under the LIA used by Kiknadze and Mamaladze [20] the vortex length remains constant [22], whereas the quantity which is conserved under the exact BSL is the energy. Length and energy are proportional to each other only if the vortex filament is straight, which is not the case in our problem. Indeed, further investigation reveals that the vortex motion contains two charac-



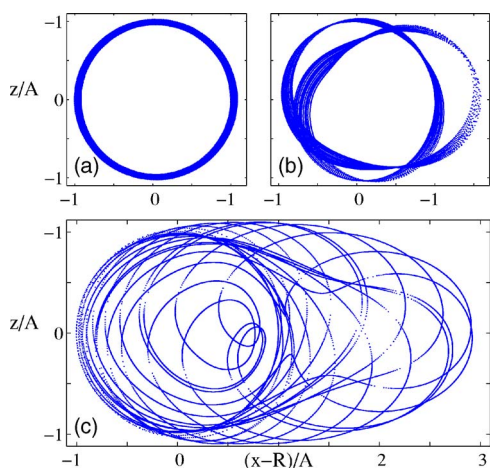


FIG. 3. (Color online) Curve drawn by the vortex at  $y=0$  plane. Here the  $z$  coordinate is the coordinate relative to the average location of the vortex and  $N=10$  in Eq. [6]. In the top left panel (a) the amplitude is  $A/R=0.05$  and in the top right panel (b)  $A/R=0.20$ . In both panels only the first 30 sec are shown. The thickness of the plotted curve arises from the chaotic motion rather than initial transient. The bottom panel (c) corresponds to  $A/R=0.50$  and we have drawn the curve for the first 90 sec. The time step between the markers is 2 ms; it is apparent that at large amplitudes the vortex is far from a sinusoidal helix and that the rotational speed at  $y=0$  plane varies significantly.

teristic frequencies. The first is the Kelvin frequency and the second is the frequency that is related to the oscillations of the vortex length and illustrated in Fig. 5. If the ratio of the two periods is rational one observes a fully periodic motion (in addition to translational motion along the  $z$  axis). At some values of the amplitudes which we calculated, this condition is almost satisfied. At higher values of amplitude one observes that the average radius of the vortex ring oscillates, as

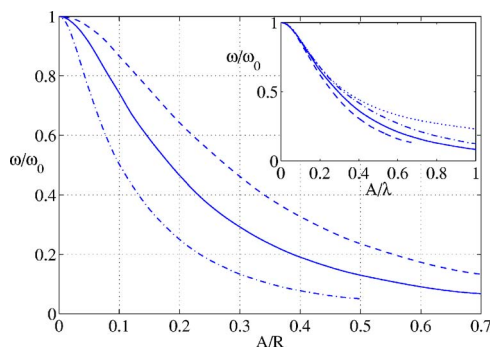


FIG. 4. (Color online) Main figure: Angular frequency of Kelvin waves  $\omega$  relative to the value  $\omega_0$  obtained in the small amplitude limit  $A/R=0.001$  and presented as a function of the wave amplitude  $A/R$ . The dashed line is for  $N=6$ , the solid line for  $N=10$ , and the dash-dotted line for  $N=20$ . The inset shows the same when plotted as a function of  $A/\lambda$ , where  $\lambda$  is the wavelength of the Kelvin wave. The additional dotted line is the result obtained for straight vortex when using a wavelength of 0.1 cm together with periodic boundary conditions and using 25 periods above and below to numerically determine the vortex motion.

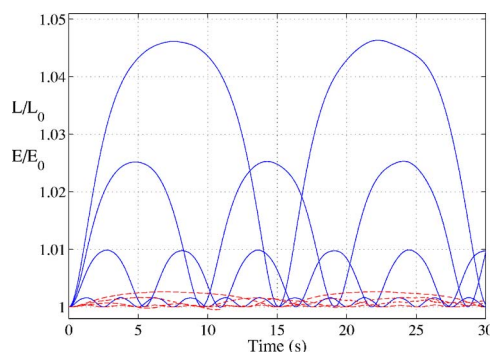


FIG. 5. (Color online) The observed vortex length compared with the initial length  $L_0=2\pi\sqrt{R^2+N^2A^2}$  is illustrated by solid (blue) lines and plotted as function of time in case of  $N=10$ . For comparison, the dashed (red) lines show the fluctuations in energy that are due to numerical errors and which can be reduced by increasing the space resolution. With increasing amplitude of oscillations the parameters for  $A/R$  shown are 0.20, 0.30, 0.40, and 0.50.

shown in Fig. 1. These variations in the total length were observed but not discussed in a recent calculation of the motion of vortex rings using the NLSE model [28].

The accuracy of our numerical method is tested by calculating the energy of the vortex ring. At zero temperature, without any dissipation, the energy (and the momentum) should remain constant. This condition can be quite well satisfied in our calculations. We do get some small oscillations in energy, as seen in Fig. 5, but we have checked that by increasing the space resolution we can reduce them at will, whereas the oscillations in length are independent of the numerical resolution.

#### IV. CONCLUSION

It is well known that a circular vortex ring has a translational velocity which arises from its own curvature (the smaller the radius  $R$  of the ring, the faster the ring travels). Using the exact Biot-Savart law, we have analyzed the motion of a vortex ring perturbed by Kelvin waves of finite amplitude. We have found that the translational velocity of the perturbed ring decreases with increasing amplitude; at some critical amplitude the velocity becomes zero, that is, the vortex ring hovers like a helicopter. A further increase of the amplitude changes the sign of the translational velocity, that is, the vortex ring moves backward. Our finding confirms preliminary results obtained by Kiknadze and Mamaladze using the local induction approximation [20].

This remarkable effect is due to the tilt of the plane of the Kelvin waves which induce motion in the “wrong” direction. The magnitude of the tilt oscillates and what results is a wobbly translational motion in the backward direction. We have also found that the frequency of the Kelvin wave decreases with increasing amplitude and that the total length of the perturbed vortex ring oscillates with time. This oscillation in vortex length is related to the oscillation of the tilt angle.

Time of flight measurements of large, electrically charged,

perturbed vortex rings in  $^4\text{He}$  could easily detect the decreased translational velocity. Another context in which the effect can be studied is Bose-Einstein condensation in ultracold atomic gases, which allow simple visualization of individual vortex structures. For these systems, however, it would be necessary to assess the effect of the nonhomogeneity of the superfluid.

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